Characterizing Quantum Properties of a Measurement Apparatus

Experimental Illustration with single-photon detectors

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Content of this talk

1. Why and how measuring measurement?
2. States and propositions
3. Preparations and Measurements
4. How to study an apparatus?
5. Illustration with single-photon detectors.
6. Conclusions and perspectives
Why Measuring Measurement?

Quantum State Tomography (QST)

The reconstruction of a quantum state needs a precise knowledge of measurements since it is based on Born’s rule

\[ \Pr(n \mid \hat{\rho}_m) = \text{Tr}\left\{ \hat{\rho}_m \hat{\Pi}_n \right\} \]

Finding the density matrix giving the probabilities which are the closest to those measured

\[ \hat{\rho}_m = \hat{\rho}_m^\dagger \geq 0 \quad \text{Tr}\left\{ \hat{\rho}_m \right\} = 1 \]
How Measuring Measurement?

**Quantum Detector Tomography (QDT)**

The behavior of an apparatus is probed by a set of known states. The probabilities are calculated as:

\[ \Pr(n \mid \hat{\rho}_m) = Tr\{\hat{\rho}_m \hat{\Pi}_n\} \]

- **Probe states** \( \{\hat{\rho}_m\} \)
- **Measurement apparatus** \( \{\hat{\Pi}_n\} \)

Finding the POVMs giving the probabilities which are closest to those measured:

\[ \hat{\Pi}_n = \hat{\Pi}_n^\dagger \geq 0 \quad \sum_n \hat{\Pi}_n = \hat{1} \]

- J.S. Lunden et al., Nature Phys. 5, 27-30 (March 2009)
“Chicken or the egg” dilemma with states and measurements.

QDT results are not yet exploited for having relevant properties about the measurements ...

This is the aim of this talk !
States and Propositions

Back to the mathematical foundations of quantum theory ...

The expression of probabilities on the Hilbert space is given by the recent generalization of Gleason’s theorem (2003) based on

- general requirements about the probabilities,
- the mathematical structure of the Hilbert space

**Statement**: any system is described by a density operator allowing predictions about any property of the system.

A property about the system is a precise value for a given observable.

Example:

\[ P_n : \text{the light pulse contains exactly } n \text{ photons} \]

The proposition operator is:

\[ \hat{P}_n = |n\rangle\langle n| \]

from an exhaustive set of propositions:

\[ \sum_n \hat{P}_n = \hat{1} \]
A **proposition** can also be represented by a **hermitian and positive operator**

\[ \hat{P}_n = \hat{P}_n^\dagger \geq 0 \]

The probability of checking such a property should satisfy the following conditions:

1. $0 \leq \Pr(n) \leq 1$ for any proposition $P_n$.

2. $\sum_n \Pr(n) = 1$ for any exhaustive set of propositions such that $\sum_n \hat{P}_n = \hat{I}$.

3. $\Pr(n_1 \text{ or } n_2 \text{ or } ...) = \Pr(n_1) + \Pr(n_2) + ...$ for any non-exhaustive set of propositions such that $\hat{P}_{n_1} + \hat{P}_{n_2} + ... \leq \hat{I}$. 
For a system needing predictions (Hilbert space of dimension $D \geq 2$), the probability is given by:

$$\Pr(n) = Tr\left\{ \hat{\rho} \hat{P}_n \right\}$$

$$\hat{\rho} = \hat{\rho}^\dagger, \quad \hat{\rho} \geq 0, \quad Tr\{\hat{\rho}\} = 1 \rightarrow \hat{\rho} = \sum_k p_k |\psi_k\rangle \langle \psi_k|$$

This is the state of the system and the probabilities are conditioned on this state of knowledge.

$$\Pr(n|\hat{\rho}) = Tr\left\{ \hat{\rho} \hat{P}_n \right\}$$
We can only make predictions about the choices “m” and the results “n”.

- Two approaches in this game: predictive about measurement results and retrodictive about state preparations.

- Each approach needs a quantum state and propositions (Gleason’s theorem).
### Preparations and Measurements

**Predictive approach**

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<tr>
<th>State</th>
<th>Prepared (post-preparation) state</th>
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<tr>
<td></td>
<td>$\hat{\rho}_m$</td>
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<td>$\hat{U}(t)$</td>
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**Predictions about measurement results**
Preparations and Measurements

Retrodictive approach

<table>
<thead>
<tr>
<th>Time-evolution</th>
<th>( \hat{U}^\dagger (t) )</th>
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<td>Preparations</td>
<td>( \hat{\Theta}_m )</td>
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<tr>
<td>State</td>
<td>( \hat{\rho}_n^{\text{retr}} )</td>
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Predictions about state preparations
## Preparations and Measurements

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<tr>
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<td>Retrodicted (pre-measurement) state</td>
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<td>$\hat{\rho}_{n^{retr}}$</td>
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<tr>
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<td>$\hat{\Theta}_m$</td>
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<td><strong>Successful test</strong></td>
<td>$\hat{\rho}_{n^{retr}} = \frac{\hat{n}_n}{Tr{\hat{n}_n}}$</td>
<td>$\hat{\rho}_m = \frac{\hat{\Theta}_m}{Tr{\hat{\Theta}_m}}$</td>
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How to study an apparatus?

- Measurement apparatus is a **physical implementation of tests checking propositions (POVMs)**

  **How characterizing an apparatus?**

  - By reconstructing the pre-measurement states retrodicted from its responses with a new QST: **retroQST**
How to study an apparatus?

The properties of a measurement are those of its retrodicted pre-measurement state.

\[ \hat{\rho}_{n}^{\text{retr}} = \frac{\hat{\Pi}_{n}}{Tr\{\hat{\Pi}_{n}\}} \]

\[
\Pr(m|n) = Tr\{\hat{\rho}_{n}^{\text{retr}} \hat{\Theta}_{m}\}, \quad \sum_{m} \hat{\Theta}_{m} = \hat{1}
\]
Non-classicality of a measurement

It corresponds to the non-classicality of its pre-measurement state

Illustration on the conditional preparation of non-classical states of light

Projection rule:

\[
\hat{\rho}_{A,\text{cond}}^{[n]} = \frac{1}{P_n} Tr_B \left\{ \hat{\rho}_{AB} \hat{1}_A \otimes \hat{\Pi}_n \right\}
\]

\[
W_{A,\text{cond}}^{[n]} (\alpha_A) = N \int d^2 \alpha_B W_{AB} (\alpha_A, \alpha_B) W_{B,\text{retr}}^{[n]} (\alpha_B)
\]

Negativity in the W-representation of the pre-measurement state is necessary for preparing a non-classical state with this signature.
Projectivity of a measurement

Purity of its pre-measurement state:

\[ \pi_n = Tr \left( \left( \hat{\rho}^{\text{retr}}_n \right)^2 \right) \]

Measurement is **projective** for \( \pi_n = 1 \)

\[ \pi_n = 1 \rightarrow \hat{\rho}^{\text{retr}}_n = \frac{\hat{\Pi}_n}{Tr\{\hat{\Pi}_n\}} = |\psi_n\rangle \langle \psi_n| \]

...But a projective measurement may be **non-ideal in the predictive approach**

\[ \Pr(n|\psi_n) = Tr \left( |\psi_n\rangle \langle \psi_n| \hat{\Pi}_n \right) = Tr \{ \hat{\Pi}_n \} = \eta_n \]

This is the **detection efficiency of the target state** \( |\psi_n\rangle \)
Fidelity of a measurement

Fidelity with a projective measurement

the overlap between the pre-measurement state and a target state in which we would like checking the system

$$F_n\left(\psi_{\text{tar}}\right) = \langle \psi_{\text{tar}} | \hat{\rho}^{\text{retr}}_n | \psi_{\text{tar}} \rangle$$

In the retrodictive approach, it is a retrodictive probability

$$F_n\left(\psi_{\text{tar}}\right) = \text{Pr}\left(\psi_{\text{tar}} | n\right) = Tr\left\{ \hat{\rho}^{\text{retr}}_n \Theta_{\text{tar}} \right\}$$

$$\Theta_{\text{tar}} = | \psi_{\text{tar}} \rangle \langle \psi_{\text{tar}} |$$

When the measurement is sufficiently faithful, the most probable preparation is the target state
How to study an apparatus?

Quantum tomography of pre-measurement states
Experimental illustration with single-photon detectors

\[ f(n|m) \rightarrow f(m|n) = \frac{f(n|m)}{\sum_{m'=1}^{M} f(n|m')} \]

Bayes’ theorem:
How to study an apparatus?

Quantum tomography of pre-measurement states

We realize a MaxLike estimation of the retrodicted pre-measurement state

\[
Pr(m|n) = Tr\left\{\hat{\rho}_n \hat{\Lambda}_m\right\} \approx f(m|n)
\]

We replace the POVMs in the iterative method by the preparation operators: \textit{retroQST}.

We can also obtain the POVM (without a QDT) from the MaxLike-estimated state.

T. Amri, V. D’Auria, J. Laurat and C. Fabre, Quantum tomography of retrodicted pre-measurement states for single-photon detectors, in preparation
Experimental illustration

Results for an Avalanche PhotoDiode (APD)

Result ‘OFF’

\[ F \approx 99.5\% \]

Relative error between estimated and measured retrodicted frequencies < 1%!
Experimental illustration

Results for a Time-Multiplexed APD

Result ‘ON’

\[ F \approx 99.7\% \]

\[ n \sim 1 \]
Experimental illustration

Non-classicality of an APD measurement

$\hat{\Pi}_{\text{on}} = \hat{1} - \hat{\Pi}_{\text{off}}$

$\rightarrow W_{\text{on}}(x, p) = \frac{1}{2\pi}W_{\text{off}}(x, p)$

Lost of non-classicality under the noise influence.

$W_{\text{on}}(0, 0)$

$\eta \approx 8\%$

-0.012 0.018 0.062 0.063

$\eta \approx 12\%$

-0.011 0.002 0.050 0.056

$\eta \approx 26\%$

-0.016 -0.001 0.007 0.047

$\nu \approx 0.00 \quad \nu \approx 0.08 \quad \nu \approx 0.16 \quad \nu \approx 0.36$

$W$-representation of the POVM reconstructed from the retroQST
Conclusions and perspectives

- **Relevant quantum properties about measurements (non-classicality, projectivity, fidelity...)**


- **Real status for the retrodictive approach of quantum physics (QST of pre-measurement states ...)**

- **Designing measurement devices characterized by more exotic non-classical pre-measurement states (Schrödinger’s cat states detector, ...)**

  T. Amri et al. *Detector of Schrödinger’s Cat States of Light for Quantum Metrology*, in preparation